John decompositions: selecting a large part

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Abstract

We extend the invertibility principle of J. Bourgain and L. Tzafriri to operators acting on arbitrary decompositions $id = \sum x_j \otimes x_j$, rather than on the coordinate one. The John's decomposition brings this result to the local theory of Banach spaces. As a consequence, we get a new lemma of Dvoretzky-Rogers type, where the contact points of the unit ball with its maximal volume ellipsoid play a crucial role. This is applied to embeddings of l_∞^k into finite dimensional spaces.

1 Introduction

The aim of this paper is to find a part of a John decomposition on which a given nontrivial operator is invertible in a certain sense, and to apply this to the study of contact points of convex bodies.

John decomposition of identity is by now a classical tool in the local theory of Banach spaces. Suppose $X = (\mathbf{R}^n, \|\cdot\|)$ is a Banach space whose ellipsoid of maximal volume contained in $B(X)$ coincides with the unit euclidean ball. The John decomposition of the identity operator on X is

$$(1) \qquad id = \sum_{j=1}^{m} x_j \otimes x_j,$$

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where $x_j/\|x_j\|_X$ are some contact points of the surfaces of $B(X)$ and the unit euclidean ball. This celebrated theorem of F. John has been used extensively over past 30 years. Recently it was interpreted as an isotropic condition [G-M], and was generalized to a non-convex case in [G-P-T].

It is important to know good parts of John decomposition, as it provides a good set of contact points of X, which can be useful for understanding the geometrical structure of X, see [R2]. In the present paper we find a part of a decomposition (1) which preserves the orthogonal structure under action of a given linear operator T. More precisely, if $\|T\|_{2\to 2} = 1$ then there exists a subset of indices σ of cardinality $|\sigma| \geq (1-\varepsilon)\|T\|_{\mathrm{HS}}^2$ such that the system $(Tx_j)_{j\in\sigma}$ is $C(\varepsilon)$-equivalent to an orthogonal basis in Hilbert space. This result is non-trivial even for the coordinate decomposition $id = \sum e_j \otimes e_j$. In this case it generalizes the principle of restricted invertibility proved by J. Bourgain and L. Tzafriri [B-Tz]. They considered only operators T for which all norms $\|Te_j\|_2$ are well bounded below and proved the principle with some fixed $0 < \varepsilon < 1$.

T being an orthogonal projection, we derive a new lemma of Dvoretzky-Rogers type. Suppose P is an orthogonal projection in X with $\mathrm{rank}P = k$. Then for any $\kappa < k$ there are contact points $x_1, \ldots, x_\kappa$ such that setting $z_j = Px_j/\|Px_j\|_2$ we have

- the system (z_j) is $C(\kappa/k)$-equivalent in l_2-norm to the canonical basis of l_2^κ;

- $\|z_j\|_X \geq c\sqrt{\frac{k-\kappa}{n}}$ for all j.

To put the result in other words, the orthonormal system in Z guaranteed by the classical Dvoretzky-Rogers Lemma is essentially the normalized projections of contact points of X. Moreover, the result holds for selfadjoint operators as well as for projections, the Hilbert-Schmidt norm substituting $\mathrm{rank}P$. For a general operator T the best lower bound for $\|Tx_j\|_X$ is equivalent to $\frac{1}{n}|\mathrm{trace}T|$.

T being the identity operator, we obtain a set of $k > (1-\varepsilon)n$ contact points of X which is $C(\varepsilon)$-equivalent in l_2-norm to the canonical basis in l_2^k. This settles an isomorphic version of a problem of N. Tomczak-Jaegermann ([T-J], p.127), and confirms the feeling that contact points are always distributed fairly uniformly on the surface of the maximal volume ellipsoid

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(see [B1]). Besides, this yields the proportional Dvoretzky-Rogers factorization (with constant $C(\varepsilon) = \varepsilon^{c \log \varepsilon}$, which is however not the best known estimate).

The Dvoretzky-Rogers Lemma is proved useful in study of subspaces of X well isomorphic to l_∞^k. The use of the refined Dvoretzky-Rogers lemma above improves a "Gaussian" version of Alon-Milman-Talagrand Theorem about l_∞^k-subspaces of X. Let P be an orthogonal projection in X with $\mathrm{rank} P = k$. Then there exists a subspace $Z \subset X$ which is M-isomorphic to l_∞^m for $m \geq ck/\sqrt{n}$ and $M = c\sqrt{\frac{n}{k}}\ell(P)$. The subspace Z is canonically spanned by the projections of m contact points x_j. Moreover, the norm on Z is M-equivalent to $|||z||| = \max_{j \leq m} |\langle z, x_j \rangle|$. This improves the estimates obtained by M. Rudelson in [R1], and also provides information about the position of Z in X. Besides, this yields a refinement of M. Rudelson's result about l_∞^k-subspaces in spaces with large volume ratio. If $\mathrm{vr}(X) \geq a\sqrt{n}$ then X has a subspace Z of dimension $m \geq C_1(a)\sqrt{n}$ which is $C_2(a) \log n$-isomorphic to l_∞^m.

The extraction results about John decompositions can be reformulated in the language of frames. Suppose we are given a tight frame (x_j) in Hilbert space H, and a norm-one linear operator $T : H \to H$. Then there is a subsequence $(Tx_j)_{j \in \sigma}$ with $|\sigma| \geq (1 - \varepsilon)\|T\|_{\mathrm{HS}}^2$ which is $C(\varepsilon)$-equivalent to an orthogonal basis in Hilbert space. This theorem again can be interpreted as an extension of the invertibility principle. It also generalizes results of P. Casazza [C2] and the author [V], who worked with the identity operator $T = id$.

The rest of the paper is organized as follows. In §2 we recall some basic tools used later. The extraction result about John decompositions, as well as some modifications, is proved in §3. Its relation to the principle of restricted invertibility and infinite-dimensional analogs are discussed in §4. Dvoretzky-Rogers type lemmas are derived from these results in §5. They help to understand structure of the set of contact points. Applications to l_∞^k-subspaces of a finite dimensional space are given in §6. Finally, in §7 we discuss a relation of these results to the theory of frames in Hilbert space.

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2 Preliminaries

We denote by c, c_1, c_2 absolute constants, and by $C(t), C_1(t), C_2(t)$ constants which depend on the parameter t only. The values of these constants may differ from line to line. The canonical vectors in $\mathbf{R}^n$ are denoted by e_j.

A sequence of vectors (x_j) in a Banach space is called K-*Hilbertian* if $\|\sum a_j x_j\| \leq K(\sum |a_j|^2)^{1/2}$ for any finite set of scalars (a_j). Similarly, (x_j) is called K-*Besselian* if $K\|\sum a_j x_j\| \geq (\sum |a_j|^2)^{1/2}$ for any finite set of scalars (a_j). Suppose we are given two sequences (x_j) and (y_j) in Banach spaces X and Y respectively. The sequences (x_j) and (y_j) are called K-*equivalent* if there exist constants K_1 and K_2 with $K_1 K_2 \leq K$ such that for any finite sequence of scalars (a_j)

$$K_1^{-1}\|\sum a_j y_j\|_Y \leq \|\sum a_j x_j\|_X \leq K_2\|\sum a_j y_j\|_Y.$$

In other words, the linear operator $T : \operatorname{span}(x_j) \to \operatorname{span}(y_j)$ defined as $Tx_j = y_j$ for all j is a K-isomorphism: $\|T\|\|T^{-1}\| \leq K$.

Here and in the next section we work in a Hilbert space H whose scalar product is denoted by $\langle \cdot, \cdot \rangle$, and the norm by $\| \cdot \|$. First we observe that the Hilbert-Schmidt norm of an operator on H can be computed on the elements of certain decompositions of identity.

Lemma 2.1 *Let* $id = \sum x_j \otimes x_j$ *be a decomposition on a Hilbert space H, and $T : H \to H$ be a linear operator. Then*

$$\|T\|_{\mathrm{HS}}^2 = \sum \|Tx_j\|^2.$$

Proof. It is enough to write

$$\sum Tx_j \otimes Tx_j = TT^* = \sum Te_j \otimes Te_j$$

and to take traces. ∎

As an immediate consequence we have

Corollary 2.2 *Let* $id = \sum x_j \otimes x_j$ *be a decomposition on a Hilbert space H. Then*

$$\sum \|x_j\|^2 = \dim H.$$

Lemma 2.3 *Let* $id = \sum x_j \otimes x_j$ *be a decomposition on a Hilbert space. Then the system* (x_j) *is* 1-*Hilbertian.*

Proof. Notice that for every vector x

$$\|x\|^2 = \langle x, x \rangle = \Big\langle \sum \langle x_j, x \rangle x_j, x \Big\rangle = \sum |\langle x_j, x \rangle|^2.$$

Thus $\|\sum x_j \otimes e_j\| = 1$, and by duality $\|\sum e_j \otimes x_j\| = 1$. This yields that (x_j) is 1-Hilbertian. ∎

The starting point of this paper is the principle of restricted invertibility proved by J. Bourgain and L. Tzafriri [B-Tz].

Theorem 2.4 *(J. Bourgain, L. Tzafriri). Let* T *be a linear operator in* l_2^n *for which* $\|Te_j\| = 1$, $j = 1, \ldots, n$. *Then there exists a subset* $\sigma \subset \{1, \ldots, n\}$ *of cardinality* $|\sigma| \geq c_1 n / \|T\|^2$ *such that*

$$\Big\| \sum_{j \in \sigma} a_j Te_j \Big\| \geq c \Big(\sum_{j \in \sigma} |a_j|^2 \Big)^{1/2}$$

for any choice of scalars (a_j).

The invertibility principle will be used together with the following restriction theorem. It can easily be recovered from A. Kashin's and L. Tzafriri's paper [K-Tz], see the proofs of Theorem 1 and Corollary 2 there.

Theorem 2.5 *(A. Kashin, L. Tzafriri). Let* A *be a norm-one linear operator in* l_2^m. *Fix a number* λ *with* $1/m \leq \lambda \leq 1$. *Then there exists a subset* $\nu \subset \{1, \ldots, m\}$ *of cardinality* $|\nu| \geq \lambda m/4$ *such that*

$$\|P_\nu A\| \leq c \left(\sqrt{\lambda} + \frac{\|A\|_{\mathrm{HS}}}{\sqrt{m}} \right).$$

Here P_ν *denotes the coordinate projection onto* $\mathbf{R}^\nu$.

Now we introduce an elementary procedure of splitting a sequence. Given a sequence (x_j) in H, let (y_k) be any sequence of vectors in H such that for every $j = 1, 2, \ldots$

- the vectors y_k, $k \in \sigma_j$, are multiples of the vector x_j;

- $\sum_{k \in \sigma_j} \|y_k\|^2 = \|x_j\|^2$.

Then we say that (y_k) is the *splitted sequence* (x_j). Splitting allows us to make the norms of the vectors almost equal. Still the key property of a sequence, being h-Hilbertian, is preserved by splitting.

3 The main result

In this section we prove an extraction theorem which is a core of the paper.

Theorem 3.1 *Let $id = \sum x_j \otimes x_j$ be a decomposition on l_2^n, and T be a norm-one linear operator. Then for any $\varepsilon > 0$ there exists a set of indices σ of cardinality $|\sigma| \geq (1 - \varepsilon)\|T\|_{HS}^2$ such that*
   (i) the system $(Tx_j)_{j \in \sigma}$ is $C(\varepsilon)$-equivalent to an orthogonal basis in l_2^σ;
   (ii) $\|Tx_j\| \geq c\sqrt{\varepsilon}\frac{\|T\|_{HS}}{\sqrt{n}}\|x_j\|$ for all $j \in \sigma$.

Proof. Put $h = \|T\|_{HS}^2$. By an approximation one can assume that the system (x_j) is finite, so we enumerate it as $(x_j)_{j \leq m}$. Denote $y_j = Tx_j$ for all j. Splitting the system $(x_j)_{j \leq m}$ we can assume that

$$0.9\sqrt{\frac{h}{m}} \leq \|y_j\| \leq 1.1\sqrt{\frac{h}{m}} \quad \text{for all } j \leq m.$$

Let $\delta = \varepsilon/3$. Consider the set of indices satisfying (ii), i.e.

$$\tau = \left\{ j \leq m : \|y_j\| \geq 0.9\sqrt{\delta}\sqrt{\frac{h}{n}}\|x_j\| \right\}.$$

We claim that
$$(2) \qquad\qquad |\tau| \geq (1 - \delta)m.$$

Indeed, by Corollary 2.2

$$
\begin{aligned}
n = \sum_{j \leq m} \|x_j\|^2 &\geq \sum_{j \in \tau^c} \|x_j\|^2 \\
&\geq \sum_{j \in \tau^c} \left(\frac{1}{0.9\sqrt{\delta}} \sqrt{\frac{n}{h}} \|y_j\| \right)^2 \\
&\geq |\tau^c| \left(\frac{1}{0.9\sqrt{\delta}} \sqrt{\frac{n}{h}} 0.9\sqrt{\frac{h}{m}} \right)^2 = |\tau^c| \cdot \frac{n}{\delta m}.
\end{aligned}
$$

Thus $|\tau^c| \leq \delta m$, which proves (2).

Now we have to find a further subset $\sigma \subset \tau$ of cardinality $|\sigma| \geq (1 - \varepsilon)h$, such that the system $(y_j)_{j \in \sigma}$ is $C(\varepsilon)$-equivalent to an orthogonal basis in l_2^σ. The set σ will be constructed by successive iterations. On the first step $\sigma = \emptyset$. On each successive step, the remainder $h - |\sigma|$ will be reduced in a fixed proportion. So, it is enough to prove the following.

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Lemma 3.2 *Let $\sigma \subset \tau$ with $|\sigma| < (1-\varepsilon)h$ be given, and suppose the system $(y_j)_{j \in \sigma}$ is K-equivalent to an orthogonal basis in Hilbert space. Then σ can be extended in τ to a subset σ_1 so that*

(a) the system $(y_j)_{j \in \sigma_1}$ is $C(K, \varepsilon)$-equivalent to an orthogonal basis in Hilbert space;

(b) for some absolute constant $\alpha < 1$

$$h - |\sigma_1| \leq \alpha(h - |\sigma|).$$

Proof. Let P be the orthonormal projection in l_2^n onto $l_2^n \ominus \operatorname{span}(y_j)_{j \in \sigma}$ (at the first step $P = id$). First observe that by Lemma 2.1

$$
\begin{aligned}
\sum_{j \leq m} \|Py_j\|^2 &= \sum_{j \leq m} \|PTx_j\|^2 = \|PT\|_{\mathrm{HS}}^2 \\
&= h - \|(id - P)T\|_{\mathrm{HS}}^2 \\
&\geq h - \|id - P\|_{\mathrm{HS}}^2 \|T\|^2 \\
&= h - |\sigma|.
\end{aligned}
$$

Using (2), we get

$$
\begin{aligned}
\sum_{j \in \tau \setminus \sigma} \|Py_j\|^2 &= \sum_{j \in \tau} \|Py_j\|^2 \\
&\geq \sum_{j \leq m} \|Py_j\|^2 - |\tau^c| \cdot \left(1.1\sqrt{h/m}\right)^2 \\
&\geq h - |\sigma| - 1.21\delta h \\
&\geq (1 - 2\delta)h - |\sigma| =: h_0.
\end{aligned}
$$

Note that h_0 is comparable with h. Indeed, since $|\sigma| < (1-\varepsilon)h = (1-3\delta)h$, we have

(3) $$h_0 \geq \delta h.$$

Now we can split the system $(y_j)_{j \in \tau \setminus \sigma}$ so that the resulting system $(y_j')_{j \leq M}$ satisfies

(4) $$\|Py_j'\| \geq 0.9\sqrt{\frac{h_0}{M}} \quad \text{for all } j \leq M.$$

We are going to apply Kashin-Tzafriri's extraction result, Theorem 2.5. In the dual setting it can be reformulated as follows.

- Let $(x_j)_{j \leq m}$ be a 1-Hilbertian system in a Hilbert space, and put $\sum \|x_j\|^2 = h$. Fix a number λ with $1/m \leq \lambda \leq 1$. Then there exists a subset $\nu \subset \{1, \ldots, m\}$ of cardinality $|\nu| \geq \lambda m/4$ such that setting $K = \left(\sqrt{\lambda} + \sqrt{h/m}\right)^{-1}$,

$$\text{the system } (Kx_j)_{j \in \sigma} \text{ is } c\text{-Hilbertian.}$$

Apply this to the system $(y'_j)_{j \leq M}$ which is 1-Hilbertian and

$$\sum_{j \leq M} \|y'_j\|^2 \leq \sum_{j \leq m} \|y_j\|^2 = h.$$

With $\lambda = 4h/M$, we obtain a subset $\nu \subset \{1, \ldots, M\}$ of cardinality $|\nu| \geq h$ such that

(5) $\qquad$ the system $\left(\sqrt{\frac{M}{h}}y'_j\right)_{j \in \nu}$ is c-Hilbertian

(notice that we could make M large enough to have $\lambda \leq 1$ as required in Kashin-Tzafriri's Theorem). Therefore, the system $\left(\sqrt{\frac{M}{h}}Py'_j\right)_{j \in \nu}$ is c-Hilbertian, too. Moreover, by (4) and (3)

$$\left\|\sqrt{\frac{M}{h}}Py'_j\right\| \geq 0.9\sqrt{\frac{h_0}{h}} \geq 0.9\sqrt{\delta} \quad \text{for all } j \in \nu.$$

At this point we use the original invertibility principle of J. Bourgain and L. Tzafriri [B-Tz], which can be reformulated as follows.

- Let $(x_j)_{j \leq n}$ be an H-Hilbertian system in l_2 and $\|x_j\| \geq \alpha$ for all j. Then there exists a subset $\rho \subset \{1, \ldots, n\}$ of cardinality $|\rho| \geq c(\alpha/H)^2 n$ such that the system $(x_j)_{j \in \rho}$ is (c_1/α)-Besselian.

Apply this to the system $\left(\sqrt{\frac{M}{h}}Py'_j\right)_{j \in \nu}$. There exists a subset $\rho' \subset \nu$ of cardinality $|\rho'| \geq c(0.9\sqrt{\delta})^2 h = c\delta h$ such that

(6) $\qquad$ the system $(\sqrt{\frac{M}{h}}Py'_j)_{j \in \rho'}$ is $c_1/\sqrt{\delta}$-Besselian.

Recall that each vector y'_j with $j \in \rho'$ is a multiple of some vector $y_{k(j)}$ with $k(j) \in \tau \setminus \sigma$. By (6), these vectors $y_{k(j)}$ must be linearly independent. In

particular, the correspondence $j \mapsto k(j)$ is one-to-one. Consider the subset $\rho \subset \tau \setminus \sigma$ consisting of vectors

$$\rho = \{k(j) : j \in \rho'\}.$$

Now put $\sigma_1 = \sigma \cup \rho$.

We see that (b) is satisfied with $\alpha = 1 - c\delta/2$:

$$h - |\sigma_1| \le h - |\sigma| - c\delta h \le \alpha(h - |\sigma|).$$

The reason why (a) holds is that the system $(y_j)_{j \in \rho}$ is well equivalent to an orthogonal basis, and the spans of $(y_j)_{j \in \sigma}$ and $(y_j)_{j \in \rho}$ are well disjointed. To implement this idea, note that by the preseding observations (5) and (6) yield that there exist constants $(\lambda_j)_{j \in \rho}$ such that

- the system $(\lambda_j y_j)_{j \in \rho}$ is c-Hilbertian,

- the system $(\lambda_j P y_j)_{j \in \rho}$ is $c_1/\sqrt{\delta}$-Besselian.

Now it is easy to complete the proof. Fix any scalars $(a_j)_{j \in \rho}$. Then defining $\lambda_j = y_j/\|y_j\|$ for $j \in \sigma$ we have

$$
\begin{aligned}
\Big(\sum_{j \in \sigma \cup \rho} |a_j|^2 \Big)^{1/2}
&= \Big(\sum_{j \in \sigma} |a_j|^2 + \sum_{j \in \rho} |a_j|^2 \Big)^{1/2} \\
&\le (K + c_1/\sqrt{\delta}) \left(\Big\| \sum_{j \in \sigma} a_j \lambda_j y_j \Big\|^2 + \Big\| \sum_{j \in \rho} a_j \lambda_j P y_j \Big\|^2 \right)^{1/2} \\
&= (K + c_1/\sqrt{\delta}) \Big\| \sum_{j \in \sigma} a_j \lambda_j y_j + \sum_{j \in \rho} a_j \lambda_j P y_j \Big\|
\end{aligned}
$$

by orthogonality

$$\le (K + c_1/\sqrt{\delta}) \Big\| \sum_{j \in \sigma \cup \rho} a_j \lambda_j y_j \Big\|,$$

because $P y_j = y_j$ for $j \in \sigma$. This shows that the system $(\lambda_j y_j)_{j \in \sigma_1}$ is $(K + c_1/\sqrt{\delta})$-Besselian. Next,

$$
\begin{aligned}
\Big\| \sum_{j \in \sigma_1} a_j \lambda_j y_j \Big\|
&\le \Big\| \sum_{j \in \sigma} a_j \lambda_j y_j \Big\| + \Big\| \sum_{j \in \rho} a_j \lambda_j y_j \Big\| \\
&\le K \Big(\sum_{j \in \sigma} |a_j|^2 \Big)^{1/2} + c \Big(\sum_{j \in \rho} |a_j|^2 \Big)^{1/2} \\
&\le \sqrt{2}(K + c) \Big(\sum_{j \in \sigma \cup \rho} |a_j|^2 \Big)^{1/2},
\end{aligned}
$$

so the system $(\lambda_j y_j)_{j \in \sigma_1}$ is $\sqrt{2}(K + c)$-Hilbertian as well. This establishes (a) of the Lemma and completes the proof of Theorem 3.1. ∎

Let us rewrite Theorem 3.1 in a different form, which is useful in applications like Dvoretzky-Rogers type lemmas.

Theorem 3.3 *Let* $id = \sum x_j \otimes x_j$ *be a decomposition on* l_2^n, *and* T *be a norm-one linear operator,* $\|T\|_{\mathrm{HS}}^2 = h$. *Then for any integer* $\kappa < h$ *there exists a set of indices* σ *with* $|\sigma| = \kappa$ *such that*
    (i) the system $(Tx_j)_{j \in \sigma}$ *is* $C(\kappa/h)$-*equivalent to an orthogonal basis in* l_2^σ;
    (ii) $\|Tx_j\| \geq c\sqrt{\frac{h-\kappa}{n}}\|x_j\|$ *for all* $j \in \sigma$.

It is sometimes more useful to get a lower bound for $\langle x_j, Tx_j \rangle$ rather than for $\|Tx_j\|$.

Proposition 3.4 *In Theorem 3.1 statement (ii) can be replaced by*
    (ii') $|\langle x_j, Tx_j \rangle| \geq c\varepsilon \frac{|\mathrm{trace}T|}{n}\|x_j\|^2$ *for* $j \in \sigma$.

Proof. Notice that $\sum \langle x_j, Tx_j \rangle = \mathrm{trace}T$. Therefore, splitting our system $(x_j)_{j \leq m}$ we can assume that

$$(7) \qquad |\langle x_j, Tx_j \rangle| \geq 0.9 \frac{|\mathrm{trace}T|}{m} \qquad \text{for all } j.$$

Let us examine the proof of Theorem 3.1. The set τ was responsible for the lower bound of $\|Tx_j\|$. So, we replace τ by

$$\tau' = \Big\{ j \leq m : |\langle x_j, Tx_j \rangle| \geq (\varepsilon/5)\frac{|\mathrm{trace}T|}{n}\|x_j\|^2 \Big\},$$

and all we have to check is

$$(8) \qquad\qquad |\tau'| \geq (1 - \varepsilon/3)m.$$

By (7)

$$\tau' \supset \rho = \Big\{ j \leq m : \|x_j\|^2 \leq (3/\varepsilon)\frac{n}{m} \Big\}.$$

Since $\sum_{j \leq m} \|x_j\|^2 = n$, we have $|\rho| \geq (1 - \varepsilon/3)m$. This verifies (8) and allows us to finish the proof as in Theorem 3.1. ∎

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4 Principle of restricted invertibility

Our first application stems from the look at Theorem 3.1 as an extension of the "principle of restricted invertibility", Theorem 2.4, proved by J. Bourgain and L. Tzafriri. Indeed, for the coordinate decomposition $id = \sum e_j \otimes e_j$ we get

Corollary 4.1 *Let T be a norm-one linear operator in l_2. Then for any $\varepsilon > 0$ there exists a subset $\sigma \subset \{1, 2, \ldots\}$ of cardinality $|\sigma| \geq (1 - \varepsilon)\|T\|_{\mathrm{HS}}^2$ such that the sequence $(Te_j)_{j \in \sigma}$ is $C(\varepsilon)$-equivalent to an orthogonal basis in Hilbert space.*

This theorem generalizes the invertibility principle in two ways. First, instead of requiring that *all* norms $\|Te_j\|$ be large, we can assume only largeness of their average, which is the Hilbert-Schmidt norm of T. This makes the result independent of the dimension n of the space, $\|T\|_{\mathrm{HS}}^2$ being a natural substitute for the dimension n.

The second improvement is that we obtain the subset σ with the largest possible cardinality. Corollary 4.1 allows us to get $|\sigma| \geq (1 - \varepsilon)n/\|T\|^2$ in the original invertibility principle for any $0 < \varepsilon < 1$ (while the Bourgain-Tzafriri's argument proves only the existence of such ε). In some applications one really needs almost full percentage; particularly, this is important in estimating the distance to the cube, see [Sz-T].

Notice that the infinite-dimensional analogs of Theorem 3.1 and Corollary 4.1 hold, too.

Proposition 4.2 *Let $id = \sum x_j \otimes x_j$ be a decomposition on a Hilbert space, and T be a linear operator which is not Hilbert-Schmidt. Then for any $\varepsilon > 0$ there exists an infinite subset σ such that the sequence $(Tx_j)_{j \in \sigma}$ is $(1 + \varepsilon)$-equivalent to an orthogonal basis in Hilbert space.*

Proof. The subset σ is constructed by a standard induction argument, modulo the following claim:

- For any finite dimensional subspace E

$$\sup_j \mathrm{dist}(Tx_j/\|Tx_j\|, E) = 1.$$

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Assume the contrary. Denoting the orthogonal projection in l_2 onto E by P, we would have

$$\inf_j \left\| P(Tx_j/\|Tx_j\|) \right\| = \delta > 0,$$

that is

$$\|PTx_j\| \geq \delta \|Tx_j\| \quad \text{for all } j.$$

By Lemma 2.1, this yields

$$\|PT\|_{\text{HS}} \geq \delta \|T\|_{\text{HS}} = \infty.$$

But the operator PT has finite rank, thus $\|PT\|_{\text{HS}}$ must be finite. This contradiction completes the proof. ∎

An infinite dimensional analogue of Bourgain-Tzafriri's Theorem 2.4 says that, given a linear operator T in l_2 with $\|Te_j\| = 1$, $j = 1, 2, \ldots$, there exists a subset σ of $\{1, 2, \ldots\}$ with upper density $\overline{\text{dens}}\, \sigma \geq c/\|T\|^2$, such that the sequence $(Te_j)_{j \in \sigma}$ is c-Besselian [B-Tz].

As we loose the normalizing condition $\|Te_j\| = 1$, nothing can be said in general about the density of σ. Indeed, let (y_k) be the result of a splitting of a canonical basis in l_2, so that the sets σ_j from the definition of splitting satisfy $|\sigma_j| \to \infty$ as $j \to \infty$. There exist a norm-one operator T in l_2 such that $\|T\|_{\text{HS}} = \infty$ and $Te_k = y_k$, $k = 1, 2, \ldots$. However, each term y_k in the sequence (y_k) is repeated $|\sigma_j|$ times. Therefore, the upper density of any subset σ satisfying the conclusion of Corollary 4.1 must be zero.

Similarly, in some cases σ must be a sparse set with respect to the dimension. More precisely, in general the sequence $(Tx_j)_{j \in \sigma}$ spans a subspace of infinite codimension. This follows easily from a result of P. Casazza and O. Christensen discussed in Section 7.

5 Contact points and Dvoretzky-Rogers lemmas

Here we apply Theorem 3.1 to John decompositions. Let $X = (\mathbf{R}^n, \| \cdot \|)$ be a Banach space. The ellipsoid of maximal volume contained in $B(X)$ is unique and it is called *maximal volume ellipsoid* of X. Suppose the euclidean

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structure on X is chosen so that the maximal volume ellipsoid coincides with the unit euclidean ball D_n. Let us write a John decomposition on X:

$$(9) \qquad\qquad id = \sum_{j=1}^{m} x_j \otimes x_j,$$

where $x_j/\|x_j\|_X$ are some contact points of $B(X)$ with the John's ellipsoid (see [T-J], §15.3). John decompositions can be considered as a subclass in the class of all decompositions of type $id = \sum x_j \otimes x_j$. Conversely, each decomposition (9) is a John decomposition for a suitable Banach space $X = (\mathbf{R}^n, \|\cdot\|)$, whose maximal volume ellipsoid is the unit euclidean ball D_n. Actually, the norm on X is given by $\|x\|_X = \max_{j\leq m} |\langle x, \frac{x_j}{\|x_j\|_X} \rangle|$. This result goes back to F. John [J], although other proofs were found recently by K. Ball [B2] and A. Giannopoulos and V. Milman [G-M]. Therefore, working with contact points instead of decompositions $id = \sum x_j \otimes x_j$ we do not lose generality.

Recasting Theorem 3.1 in this light, we have

Corollary 5.1 *Let X be an n-dimensional Banach space whose maximal volume ellipsoid is the unit euclidean ball. Let T be a linear operator with $\|T\|_{2\to 2} \leq 1$. Then for any $\varepsilon > 0$ there are contact points $x_1, \ldots, x_k$ with $k \geq (1-\varepsilon)\|T\|_{\mathrm{HS}}^2$ such that the system $(Tx_j)_{j\leq k}$ is $C(\varepsilon)$-equivalent in l_2-norm to the canonical basis of l_2^k.*

Moreover, the norms $\|Tx_j\|_2$ are well bounded below:

$$\|Tx_j\| \geq C_1(\varepsilon)\frac{\|T\|_{\mathrm{HS}}}{\sqrt{n}} \qquad \text{for all } j \leq k.$$

T being an orthogonal projection, this result leads us to a new Dvoretzky-Rogers type lemma , which we will discuss now. Suppose X is an n-dimensional Banach space whose John's ellipsoid is the unit euclidean ball. Let Z be a k-dimensional subspace of X. Dvoretzky-Rogers Lemma states that, given a positive integer $\kappa < k$, there is an orthonormal system $(z_j)_{j\leq\kappa}$ in Z such that

$$\|z_j\|_X \geq \sqrt{\frac{k-\kappa+1}{n}} \qquad \text{for all } j \leq \kappa.$$

Let us sketch the proof. By induction, it is enough to find *one* vector z in Z such that $\|z\|_2 = 1$ and $\|z\|_X \geq \sqrt{\frac{k}{n}}$ (then substitute Z by $Z \ominus \mathrm{span}(z)$ and repeat the argument). By duality, this is equivalent to finding a functional $x^* \in B(X^*)$ with $\|Px^*\|_2 \geq \sqrt{\frac{k}{n}}$, where P is the orthogonal projection onto Z. Let $id_{X^*} = \sum \lambda_j x_j^* \otimes x_j^*$ be a John decomposition in X^*, that is $\sum \lambda_j = n$ and x_j^* are contact points of $B(X^*)$. Then $P = \sum \lambda_j Px_j^* \otimes Px_j^*$. Taking the trace, we get $k = \sum \lambda_j \|Px_j^*\|_2^2$. So, there is a j such that $\|Px_j^*\|_2^2 \geq \sqrt{\frac{k}{n}}$. This completes the proof.

However, this argument, as well as other known proofs of the Dvoretzky-Rogers Lemma, only establish the existence of the vectors z_j. In contrast to that, the argument based on Theorem 3.1 provides information about their position.

Theorem 5.2 *Let X be an n-dimensional Banach space whose maximal volume ellipsoid is the unit euclidean ball. Let P be an orthogonal projection, $\mathrm{rank}P = k$. Then for any positive integer $\kappa < k$ there are contact points $x_1, \ldots, x_\kappa$ such that setting $z_j = Px_j / \|Px_j\|_2$ we have*
    (i) the system (z_j) is $C(\kappa/k)$-equivalent in l_2-norm to the canonical basis of l_2^κ;
    (ii) $\|z_j\|_X \geq c\sqrt{\frac{k-\kappa}{n}}$ for all j.

Proof. Note that $\|P\|_{\mathrm{HS}} = \sqrt{k}$ and apply Theorem 3.3 to a John decomposition on X. This gives us contact points $x_1, \ldots, x_\kappa$ such that (i) is satisfied, and

$$\|Px_j\|_2 \geq c\sqrt{\frac{k-\kappa}{n}} \quad \text{for every } j \leq m.$$

It remains to note that for every $j \leq m$

$$\|Px_j\|_X \geq \frac{\langle Px_j, x_j \rangle}{\|x_j\|_{X^*}} = \langle Px_j, x_j \rangle = \|Px_j\|_2^2,$$

hence

$$\|z_j\|_X \geq \|Px_j\|_2 \quad \text{for every } j \leq m.$$

This completes the proof. ∎

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A natural question is whether Theorem 5.2 can be extended for arbitrary operator T, $\|T\|_{2\to 2} = 1$, with k substituted by $\|T\|_{\mathrm{HS}}^2$. The answer is negative. Indeed, let $n = 2^m$ for a positive integer m, and denote by W_m the Walsh $n \times n$ matrix. Consider the operator $T = n^{-1/2}W_m$ acting in the space $X = l_\infty^n$. All contact points (x_j) of X are simply the coordinate vectors. However, denoting $z_j = Tx_j/\|Tx_j\|_2$ we have for any j

$$\|z_j\|_X = \|Tx_j\|_X/\|Tx_j\|_2 = n^{-1/2}.$$

This shows that (ii) in Theorem 5.2 would fail for the operator T.

Still, a lower bound for $\|\cdot\|_X$-norm exists and is equivalent to $\frac{1}{n}|\mathrm{trace}T|$.

Proposition 5.3 *Let X be an n-dimensional Banach space whose maximal volume ellipsoid is the unit euclidean ball. Let T be an operator with $\|T\|_{2\to 2} \le 1$. Then for any $\varepsilon > 0$ there are $k > (1-\varepsilon)n$ contact points $x_1, \ldots, x_\kappa$ such that*
  (i) the system (Tx_j) is $C(\varepsilon)$-equivalent in l_2-norm to the canonical basis of l_2^κ;
  (ii) $\|Tx_j\|_X \ge c\varepsilon \frac{|\mathrm{trace}T|}{n}$.

Proof. The argument is similar to that of Theorem 5.2; one only uses Proposition 3.4 instead of Theorem 3.1. ∎

The estimate in (ii) is essentially sharp. Indeed, for any positive integers $k \le n$ one can construct an orthogonal projection P in $\mathbf{R}^n$ with $\mathrm{rank}P = k$, and such that

$$\|Pe_j\|_2 = \sqrt{\frac{k}{n}} \quad \text{for all } j \le n.$$

Notice that $\|P\|_{1\to 2} = \sqrt{\frac{k}{n}}$, therefore $\|P\|_{2\to\infty} = \sqrt{\frac{k}{n}}$. Thus $\|P\|_{1\to\infty} \le \frac{k}{n}$. Now consider the space $X = l_\infty^n$ and the operator $T = P$ on it. The only contact points (x_j) of X are the coordinate vectors (e_j). Then for all j

$$\|Tx_j\|_X = \|Pe_j\|_\infty \le \frac{k}{n} = \frac{|\mathrm{trace}T|}{n}.$$

This shows that the lower bound in (ii) is essentially sharp.

Finally, there is a class of operators for which Theorem 5.2 itself can be extended: selfadjoint operators. So, the desired general result for arbitrary

operator can be obtained if we allow a suitable rotation of (z_j). There is always a unitary operator (coming from the polar decomposition of T) which sends the vectors (z_j) to vectors of a good $\|\cdot\|_X$-norm.

Theorem 5.4 *Let X be an n-dimensional Banach space whose maximal volume ellipsoid is the unit euclidean ball. Let T be an operator with $\|T\|_{2\to 2} \leq 1$, and put $\|T\|_{HS}^2 = h$. Then for any positive integer $\kappa < h$ there are contact points $x_1, \ldots, x_\kappa$ such that setting $z_j = |T|x_j/\|Tx_j\|_2$ we have*

(i) the system (z_j) is $C(\kappa/h)$-equivalent in l_2-norm to the canonical basis of l_2^κ;

(ii) $\|z_j\|_X \geq c\sqrt{\frac{h-\kappa}{n}}$ for all j.

Proof. Let $T = U|T|$ be the polar decomposition of T, where $|T| = (T^*T)^{1/2}$ is a positive selfadjoint operator and U is a partial isometry on l_2^n. Apply Theorem 3.3 to the operator $|T|$ and a John decomposition on X. As before, this gives us contact points $x_1, \ldots, x_\kappa$ such that (i) is satisfied, and

$$\left\||T|x_j\right\|_2 \geq c\sqrt{\frac{h-\kappa}{n}} \quad \text{for every } j \leq m.$$

From diagonalization of $|T|$ it follows that $\left\||T|^{1/2}\right\|_{2\to 2} = \|T\|_{2\to 2}^{1/2} \leq 1$. Therefore we can bound for every $j \leq m$

$$
\begin{aligned}
\left\||T|x_j\right\|_X &\geq \frac{\langle |T|x_j, x_j\rangle}{\|x_j\|_{X^*}} = \left\||T|^{1/2}x_j\right\|_2^2 \\
&\geq \left\||T|x_j\right\|_2^2 = \left\|Tx_j\right\|_2^2.
\end{aligned}
$$

Hence

$$\|z_j\|_X \geq \|Tx_j\|_2 \quad \text{for every } j \leq m.$$

This completes the proof. ∎

Now we turn to a particular case when T is the identity operator, which also happens to be interesting. We clearly have

Corollary 5.5 *Let X be an n-dimensional Banach space whose maximal volume ellipsoid is the unit euclidean ball. Then for any $\varepsilon > 0$ there is a set of $k > (1-\varepsilon)n$ contact points which is $C(\varepsilon)$-equivalent to the canonical vector basis of l_2^k.*

This result is related to another variant of the classical Dvoretzky-Rogers Lemma, which establishes the existence of contact points whose distance to a certain orthonormal basis is controlled ([T-J], Theorem 15.7). More precisely, there exist contact points $x_1, \ldots, x_n$ and an orthonormal basis $h_1, \ldots, h_n$ such that

$$\|x_j - h_j\|_2 \leq 2\Big(1 - \sqrt{\frac{n-i+1}{n}}\Big) \quad \text{for } i \leq n.$$

However, this estimate is to crude to assure that a fixed proportion of the system $(x_j)_{j \leq n}$ is equivalent in l_2-norm to an orthonormal system. Such an isomorphism can be established only for $c\sqrt{n}$ contact points.

Using this argument, it is proved that for $k = [\sqrt{n}/4]$ there exist orthonormal vectors $x_1, \ldots, x_k$ in $(E, \|\cdot\|_2)$ on which all three norms $\|\cdot\|$, $\|\cdot\|_2$, and $\|\cdot\|_*$ differ by the factor 2 at most ([T-J], p.127). It has been an open problem to make k proportional to n. By Corollary 5.5 we actually have $k \geq (1-\varepsilon)n$ and make all three norms $\|\cdot\|$, $\|\cdot\|_2$, and $\|\cdot\|_*$ equal to 1 on our sequence (paying however in exact orthogonality).

By duality, Corollary 5.5 holds also for the ellipsoid of minimal volume containing $B(X)$ instead of the maximal volume ellipsoid. This variant of Corollary 5.5 yields also a proportional Dvoretzky-Rogers factorization result from [B-Sz]. Namely, given an n-dimensional Banach space X and $\varepsilon > 0$, there is a $k > (1-\varepsilon)n$ such that the formal identity $id : l_2^k \to l_\infty^k$ can be written as $id = \alpha \cdot \beta$ for some $\beta : l_2^k \to X$, $\alpha : X \to l_\infty^k$, with $\|\alpha\|\|\beta\| \leq C(\varepsilon)$. This can be obtained by duality from the factorization of the identity on the contact points, $id : l_1^k \to X \to l_2^k$, guaranteed by Corollary 5.5.

A few comments about the dependence $C(\varepsilon)$ in Theorem 3.1. It is a challenge to find the correct a[sic] asympthotics. Indeed, by an argument of S. Szarek and M. Talagrand [Sz-T] the proportional Dvoretzky-Rogers factorization above yields a non-trivial estimate on the distance from X to l_∞^n – a well known and hard problem in the local theory of normed spaces. The factorization constant $C(\varepsilon)$ lies in the heart of the computation of this distance.

The proof of Theorem 3.1 guarantees that $C(\varepsilon) \leq \varepsilon^{c \log \varepsilon}$. However, for the Dvoretzky-Rogers factorization constant $C_{\mathrm{DR}}(\varepsilon)$ much better bounds are found [G]: $C_{\mathrm{DR}}(\varepsilon) \leq c\varepsilon^{-1}$. Since $C_{\mathrm{DR}}(\varepsilon) \leq C(\varepsilon)$ and $C_{\mathrm{DR}}(\varepsilon) \to \infty$ as $\varepsilon \to 0$ (see [Sz-T]), we necessarily have $C(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

However, $C(\varepsilon) \to 1$ as $\varepsilon \to 1$. This follows directly from

Lemma 5.6 *Let a normalized sequence $(x_j)_{j \leq n}$ in Hilbert space be M-Hilbertian, and $\varepsilon > 0$. Then there is a subset $\sigma \subset \{1, \ldots, n\}$ of cardinality $|\sigma| \geq C(M, \varepsilon)n$ such that the system $(x_j)_{j \in \sigma}$ is $(1 + \varepsilon)$-equivalent to the canonical basis of l_2^{σ}.*

Proof. We can assume that the given Hilbert space is l_2^n. Define a linear operator $T : l_2^n \to l_2^n$ by

$$Te_j = x_j \quad \text{for } j \leq n.$$

Let $A = T^*T - id$. Then the matrix of A has zeros on the diagonal and $\|A\| \leq M^2 + 1$. Now, by a theorem of J. Bourgain and L. Tzafriri ([B-Tz] Theorem 1.6, see also [K-Tz]) there is a subset $\sigma \subset \{1, \ldots, n\}$ of cardinality $|\sigma| \geq C(M, \varepsilon)$ such that

$$\|P_\sigma A P_\sigma\| < \varepsilon.$$

This shows that for any sequence of scalars $(a_j)_{j \leq n}$

$$\left| \left\langle (T^*T - id) \sum_{j \in \sigma} a_j e_j, \sum_{j \in \sigma} a_j e_j \right\rangle \right| < \varepsilon,$$

hence

$$\left| \left\langle \sum_{j \in \sigma} a_j x_j, \sum_{j \in \sigma} a_j x_j \right\rangle - \sum_{j \in s} |a_j|^2 \right| < \varepsilon.$$

This clearly finishes the proof. ∎

6 Embeddings of the cube

In this section we apply Theorem 3.1 to the study of embeddings of l_∞^k into finite dimensional spaces. N. Alon and V. Milman proved that if a given normalized sequence (x_j) in a Banach space X has small Rademacher average $\mathbf{E} \| \sum \varepsilon_j x_j \|_X$, then it must contain a large subsequence well equivalent to the canonical basis of l_∞^k. Later on, M. Talagrand [T] improved this result and simplified the argument.

Theorem 6.1 *(M. Talagrand). Suppose we are given vectors $(x_j)_{j \leq n}$ in a Banach space X with $\|x_j\|_X \geq 1$. Set $M = \mathbf{E} \| \sum \varepsilon_j x_j \|_X$ and $\omega =$*

$\sup \left\{ \sum |x^*(x_j)| : x^* \in B(X^*) \right\}$. *Then there exists a subset $\sigma \subset \{1, \ldots, n\}$ of cardinality $|\sigma| \geq cn/\omega$ such that*

$$\frac{1}{2} \max_{j \in \sigma} |a_j| \leq \Big\| \sum_{j \in \sigma} a_j x_j \Big\|_X \leq 4M \max_{j \in \sigma} |a_j|$$

for any choice of scalars (a_j).

A few years earlier, M. Rudelson obtained in [R1] a "Gaussian" version of this theorem. Recall that the ℓ-norm of an operator $u : l_2^n \to X$ is defined as

$$\ell(u)^2 = \int \|ux\|_X^2 \, d\gamma_n(x),$$

where γ_n is the canonical Gaussian measure on $\mathbf{R}^n$. We will sometimes write $\ell(X)$ instead of $\ell(id_X)$.

Suppose X is an n-dimensional Banach space whose maximal volume ellipsoid is the unit euclidean ball. Let P be an orthogonal projection in X and set $k = \mathrm{rank}P$, $a = k/n$. The result of M. Rudelson states that there is a subspace $Z \subset P(X)$ of dimension $m \geq C_1(a)\frac{\sqrt{n}}{\ell(P)}$ which is $C_2(a)\ell(P)$-isomorphic to l_∞^m.

We will remove $\ell(P)$ from the estimate on the dimension. Further, it will be shown that Z is canonically spanned by the projections of some contact points of X. In particular, the norm on Z is well equivalent to $\max_{j \leq m} |\langle z, x_j \rangle|$, where x_j are contact points. This yields automatically that Z is well complemented by the orthogonal projection. Moreover, the dependence on a will be improved.

Theorem 6.2 *Let X be an n-dimensional Banach space whose maximal volume ellipsoid is $B(l_2^n)$. Let P be an orthogonal projection, $\mathrm{rank}P = k$. Then there are contact points $(x_j)_{j \leq m}$ with $m \geq c_1 k/\sqrt{n}$ such that*

$$\max_{j \leq m} |\langle x, x_j \rangle| \leq \|x\|_X \leq c\sqrt{\frac{n}{k}} \ell(P) \max_{j \leq m} |\langle x, x_j \rangle|$$

for every x in $Z = \mathrm{span}(Px_j)_{j \leq m}$.

A particular case $k = n$ is also interesting: we get a sequence of $m \geq c_1\sqrt{n}$ contact points which is $c\ell(X)$-equivalent to the canonical vector basis

of l_∞^m. Let us prove this latter fact separately. By Corollary 5.5, there exists a set of contact points $(x_j)_{j \le m'}$, $m' \ge n/2$, which is c-equivalent in l_2-norm to the canonical basis of $l_2^{m'}$. Let (g_j) be independent standard Gaussian random variables. To apply Talagrand's Theorem 6.1, we bound

$$M = \mathbf{E}\|\sum \varepsilon_j x_j\|_X \le c\mathbf{E}\|\sum g_j x_j\|_X \le cc_2\mathbf{E}\|\sum g_j e_j\|_X$$

by the ideal property of the ℓ-norm, see Lemma 6.4 below. Further, for any finite system of scalars (a_j)

$$\begin{aligned}
\|\sum a_j x_j\|_X &\le \|\sum a_j x_j\|_2 \\
&\le c_2\left(\sum |a_j|^2\right)^{1/2} \le c_2\sqrt{n}\max_j |a_j|.
\end{aligned}$$

Thus

$$\omega = \sup\left\{\sum |x^*(x_j)| : x^* \in B(X^*)\right\} \le c_2\sqrt{n}.$$

Now Talagrand's Theorem 6.1 finishes the proof. ∎

The proof of Theorem 6.2 is longer, but the main idea remains to combine results of Section 5 with Talagrand's theorem. First, we need to know what vectors canonically span a large subspace Z well isomorphic to l_∞^m. They happen to be multiples of some contact points $(x_j)_{j \le m}$. More precisely, we have

Proposition 6.3 *Under the assumptions of Theorem 6.2, the system $\left(\frac{Px_j}{\|Px_j\|_2}\right)_{j \le m}$ is $c\sqrt{\frac{n}{k}}\ell(P)$-equivalent to the canonical basis of l_∞^m.*

To prove this, let $(x_j)_{j \le k'}$, $k' \ge k/2$, be the contact points provided by Theorem 5.2. Put

$$z_j = \alpha\sqrt{\frac{n}{k}}\frac{Px_j}{\|Px_j\|_2}$$

for an appropriate $\alpha > 0$ and all j. Then
  (i) the system $(z_j)_{j \le k'}$ is $(c\sqrt{\frac{n}{k}})$-equivalent in l_2-norm to the canonical basis of $l_2^{k'}$;
  (ii) $\|z_j\| \ge 1$ for all j.
For a future reference note that the proof of Theorem 5.2 gives also

(10) $$\|Px_j\|_2 \ge c_1\sqrt{\frac{n}{k}}.$$

To apply Talagrand's Theorem 6.1 to the system $(z_j)_{j \leq k'}$, recall the ideal property of the ℓ-norm (cf. [T-J], §12)

Lemma 6.4 *For any two linear operators* $A : l_2^n \to X$ *and* $B : l_2^n \to l_2^n$

$$\ell(AB) \leq \|B\|\ell(A).$$

Let $(h_j)_{j \leq k'}$ be an orthonormal basis in $\mathrm{span}(z_j)$. Now we bound

$$
\begin{aligned}
M &= \mathbf{E}\| \sum \varepsilon_j z_j \|_X \leq c\mathbf{E}\| \sum g_j z_j \|_X \\
&\leq c\sqrt{\frac{n}{k}}\mathbf{E}\| \sum g_j h_j \|_X \quad \text{by (i) and Lemma 6.4} \\
&= c\sqrt{\frac{n}{k}}\ell(P(X)) = c\sqrt{\frac{n}{k}}\ell(P).
\end{aligned}
$$

Noting (ii) above, we apply Talagrand's Theorem 6.1. This finishes the proof. ∎

To obtain Theorem 6.2, Talagrand's Theorem will be used more delicately. Its proof in [T] gives the following additional property.

Lemma 6.5 *In the situation of Theorem 6.2, suppose* $(x_j^*)_{j \leq n}$ *are functionals in* X^* *such that*

$$x_j^*(x_j) \geq 1 \quad and \quad \|x_j^*\|_{X^*} = 1 \quad for\ all\ j \leq n.$$

Then the subset σ *can be found so that*

$$\sum_{j \in \sigma \setminus \{i\}} |x_i^*(x_j)| \leq 1/2 \quad for\ all\ i \in \sigma.$$

Let us turn again to the proof of Proposition 6.3. We applied Talagrand's Theorem to the system $(x_j)_{j \leq k'}$. Note that by (10)

$$
\begin{aligned}
\langle x_j, z_j \rangle &= \alpha\sqrt{\frac{n}{k}}\frac{\langle x_j, Px_j \rangle}{\|Px_j\|_2} \\
&= \alpha\sqrt{\frac{n}{k}}\|Px_j\|_2 \geq 1 \quad for\ all\ j.
\end{aligned}
$$

Therefore we obtain a subset $\sigma \subset \{1, \ldots, k'\}$ of cardinality $|\sigma| \geq c_1 k/\sqrt{n}$ such that

- the system $(z_j)_{j \in \sigma}$ is $c\sqrt{\frac{n}{k}}\ell(P)$-equivalent to the canonical basis of l_∞^σ;

- $\sum_{j \in \sigma \setminus \{i\}} |\langle x_i, z_j \rangle| \le 1/2$ for all $j \in \sigma$.

Now fix an $x = \sum_{j \in \sigma} a_j z_j$, and let $i \in \sigma$ be such that $|a_i| = \max_{j \in \sigma} |a_j|$. Then

$$
\begin{aligned}
\max_{j \in \sigma} |\langle x, x_j \rangle| &\ge |\langle x_i, x \rangle| = \Big| \sum_{j \in \sigma} a_j \langle x_i, z_j \rangle \Big| \\
&\ge |a_j| \Big(|\langle x_i, z_i \rangle| - \sum_{j \in \sigma \setminus \{i\}} |\langle x_i, z_j \rangle| \Big) \\
&\ge \frac{1}{2} |a_i| = \frac{1}{2} \max_{j \in \sigma} |a_j| \\
&\ge \Big(c\sqrt{\frac{n}{k}} \ell(P) \Big)^{-1} \|x\|_X.
\end{aligned}
$$

This proves the second inequality in Theorem 6.2, while the first one is obvious. ■

Theorem 6.2 yields also that Z is well complemented by the orthogonal projection.

Proposition 6.6 *In the situation of Theorem 6.2, let P_Z be the orthogonal projection in X onto Z. Then*

$$
\|P_Z\| \le c\sqrt{\frac{n}{k}} \ell(P).
$$

Proof. For any $x \in X$

$$
\begin{aligned}
\|P_Z x\|_X &\le c\sqrt{\frac{n}{k}} \ell(P) \max_{j \le m} |\langle P_Z x, x_j \rangle| \\
&= c\sqrt{\frac{n}{k}} \ell(P) \max_{j \le m} |\langle x, x_j \rangle| \\
&\le c\sqrt{\frac{n}{k}} \ell(P) \|x\|_X.
\end{aligned}
$$

This completes the proof. ■

Another consequence of Theorem 6.2 is a refined isomorphic characterization of spaces with large volume ratio. Recall that the volume ratio of an n-dimensional Banach space X is defined as

$$\mathrm{vr}(X) = \min \left(\frac{\mathrm{Vol}(B_X)}{\mathrm{Vol}(E)} \right)^{1/n}$$

over all ellipsoids E contained in B_X, see [L-M], [P]. The maximal value of $\mathrm{vr}(X)$ among all n-dimensional spaces is of order $\sqrt{n}$, and the only space with maximal volume ratio is l_∞^n (K. Ball [B1]). Later on, M. Rudelson proved in [R1] that if $\mathrm{vr}(X)$ is proportional to the maximal volume ratio then X has a subspace isomorphic to l_∞^m with the isomorphism constant of order $\log n$, and such that $m \sim \frac{\sqrt{n}}{\log n}$. Using Theorem 6.2 in the M. Rudelson's proof of this result removes the parasitic factor $\log n$ from the estimate on the dimension.

Theorem 6.7 *Let $a > 0$, and X be an n-dimensional Banach space. If $\mathrm{vr}(X) \geq a\sqrt{n}$ then there exists a subspace Z of dimension $m \geq C_1(a)\sqrt{n}$ which is $C_2(a) \log n$-isomorphic to l_∞^m.*

7 Frames

The notion of frame goes back to the work of R. Duffin and A. Schaeffer on nonharmonic Fourier series [D-S]. A sequence (x_j) in a Hilbert space H is called a *frame* if there exist positive numbers A and B such that

$$A\|x\|^2 \leq \sum_j |\langle x, x_j \rangle|^2 \leq B\|x\|^2 \quad \text{for } x \in H.$$

The number $(B/A)^{1/2}$ is called a *constant* of the frame. We call (x_j) a *tight frame* if $A = B = 1$. For introduction to the theory of frames, its relation to wavelets and signal processing, see [B-W]. Geometric structure of frames is studied extensively in recent years, see [Ho], [A], [C-C1], [C-C2], [C2], [V].

It is known by now that finite dimensional frames are essentially the same object as John decompositions. In the equivalent theory, it is sufficient to work only with tight frames, because every frame with constant M is M-equivalent to a tight frame (cf. e.g. [Ho]). Further, one has the following

equivalence between frames and John decompositions:

$$(x_j) \text{ is a tight frame in } H \iff id_H = \sum_j x_j \otimes x_j.$$

This observation allows to interpret the results of Sections §3 and §4 as statements about frames. Theorem 3.1 yields:

Corollary 7.1 *Let (x_j) be a tight frame in Hilbert space H, and T be a norm-one linear operator in H. Then for any $\varepsilon > 0$ there exists a subset of indices σ of cardinality $|\sigma| \geq (1-\varepsilon)\|T\|_{\mathrm{HS}}^2$ such that the system $(Tx_j)_{j \in \sigma}$ is $C(\varepsilon)$-equivalent to an orthogonal basis in Hilbert space.*

For operators T which are not Hilbert-Schmidt this means that the subset σ is infinite.

Clearly, Corollary 7.1 itself generalizes the invertibility principle of J. Bourgain and L. Tzafriri. When applied to the identity operator, it yields that every tight frame in l_2^n has a subset of length $(1-\varepsilon)n$ which is $C(\varepsilon)$-equivalent to an orthogonal basis in Hilbert space. This result was proved in [V] as a generalization of P. Casazza's theorem [C2].

Notice that one necessarily has $C(\varepsilon) \to \infty$ as $\varepsilon \to 1$, as explained in Section §5. An infinite dimensional analog of this phenomenon holds, too. A frame may not in general contain a complete subsequence equivalent to an orthogonal basis. The counterexample was found by P. Casazza and O. Christensen in [C-C2], see also [V].

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